WWU Graduate Qualifying Exam, Spring 2008

You may use calculators for this exam. Be advised however that every question can be answered without the use of a calculator and more than likely can be answered more efficiently without the use of a calculator.

1. Define \( f(t) = \int_1^t \frac{1}{s} e^{s^2} \, ds \). Find \( \frac{df}{dt} \).

2. Consider the parabola \( y = 1 - x^2 \) and the triangle made from \( y = a - bx \) (and its reflection \( y = a + bx \)) which has the property that the triangle is tangent to the parabola at the points of contact. The constants \( a \) and \( b \) are positive. Two examples are shown in the diagram.

Find the numbers \( a \) and \( b \) which minimize the area under the triangle and above the \( x \)-axis (note, you can work with just one side of the symmetric problem).

3. (a) Determine whether or not the series \( \sum_{n=1}^{\infty} \frac{\sinh n}{n^n} \) converges, and justify your answer.

(b) Determine whether or not the series \( \sum_{n=1}^{\infty} (-1)^n \tan^{-1} n \) converges, and justify your answer.

(c) Determine for what real values of \( x \) the series \( \sum_{n=0}^{\infty} \frac{(x + 2)^n}{2^{(n+1)}} \) converges and for what values it diverges. Make sure that every value of \( x \) is considered. Justify your answers.

4. Consider the differential equation \( \frac{dy}{dt} = -\frac{1}{t} y + t^\alpha \), valid for \( t \geq 1 \), and where \( \alpha \in \mathbb{R} \) is a parameter.

(a) Find the general solution to this ODE. Warning: pay attention to \( \alpha \).

(b) Consider the long-time behavior \( (t \to \infty) \) of your solution(s): show that there is a value \( \alpha^* \) such that the behavior for solutions for \( \alpha < \alpha^* \) is different from the behavior of solutions for \( \alpha > \alpha^* \). Describe these behaviors, and also for the case \( \alpha = \alpha^* \).

(c) Solve the initial value problem \( y(1) = 2 \) for the case \( \alpha = 2 \).
5. While hiking along a trail the elevation increases at a rate of \( \frac{1}{3} \) meters per meter. At a point \( P \) the “current” path then veers more up-hill to a “new” path, making an angle of \( \frac{\pi}{6} \) with the current path. The steepest direction up-hill from \( P \) makes an angle of \( \frac{\pi}{3} \) with the current path.

(a) At what rate will the elevation be increasing when you veer onto the new, more up-hill, path?

(b) What angle does the new path make with the horizontal (i.e. with \( z = \) constant in 3-space)?

(c) If the steepest direction is to the North-West (not as shown in the picture), find a vector which is perpendicular to the surface at the point \( P \) (you should take the positive \( x \)-axis to be due East and the positive \( y \)-axis to be due North).

6. Refering to the diagram to the right, you must find \( \alpha \) which minimizes the sum of the perpendicular squared distances, \( f(\alpha) = d_1^2 + d_2^2 \). The points are at \((1, 2)\) and \((4, 1)\).

(a) Your first task is to find an expression for \( f(\alpha) \). Hint: think about vector projection; the vectors \((1, \alpha)\) and/or \((-\alpha, 1)\) might be useful.

(b) Find the value of \( \alpha \) which minimizes \( f \).
7. Suppose the $n \times n$ tridiagonal matrix

$$T = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 1 & -1 & \ddots & \vdots \\
0 & -1 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & 0 & -1 & 1
\end{bmatrix}$$

has $n$ eigenvalues $\lambda_j$ with corresponding eigenvectors $\vec{v}_j$, for $j = 1, 2, \ldots, n$. Find all the eigenvalues and corresponding eigenvectors of the $n \times n$ tridiagonal matrix

$$A = \begin{bmatrix}
1 + 2\sigma & -\sigma & 0 & \cdots & 0 \\
-\sigma & 1 + 2\sigma & -\sigma & \ddots & \vdots \\
0 & -\sigma & 1 + 2\sigma & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -\sigma \\
0 & \cdots & 0 & -\sigma & 1 + 2\sigma
\end{bmatrix}$$

in terms of $\lambda_j, \vec{v}_j$ and $\sigma$ (assume $\sigma \neq 0$).

8. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation that satisfies

$$T \left( \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T \left( \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$ 

Find a vector $\vec{x}$, such that $T(\vec{x}) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

9. Using cylindrical coordinates, find the volume of the region $E$ in space specified by the inequalities $x^2 + y^2 \leq 2y$ and $0 \leq z \leq \sqrt{x^2 + y^2}$.

Hint: You might want to use the integral formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$ 

10. Find the maximum and minimum values of $f(x, y) = x^2 - 2x - y$ subject to the constraint $(x - 1)^2 + y^2 = 1$. 

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