Problem 1. Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfies \( f(0) = 2 \) and \( |f(x) - f(y)| \leq |x - y|^{5/4} \) for all real numbers \( x \) and \( y \).

(a) (4 points) Prove that \( f \) is continuous at all \( x \) using the rigorous \( \epsilon - \delta \) definition of continuity.

(b) (4 points) Prove that \( f \) is differentiable at all \( x \) using the definition of the derivative.

(c) (2 points) Compute \( \int_{3}^{6} f(y) \, dy \).

Problem 2. Let \( f(x) = \int_{0}^{g(x)} \frac{1}{\sqrt{1 + t^3}} \, dt \), where \( g(x) = \int_{0}^{\cos x} (\sin(t^2) + 1) \, dt \). Find \( f'(\pi/2) \).

Problem 3. Find the point on the parabola \( y = 1 - x^2 \) in the first quadrant at which the tangent line together with the \( x \)- and \( y \)- axes forms the triangle with smallest area.

Problem 4. Find all points \( P \) on the ellipsoid \( 2x^2 + 2y^2 + z^2 = 28 \) such that the tangent plane to the ellipsoid at \( P \) is parallel to the plane passing through the three points \( (1,3,1), (3,0,-3), \) and \( (0,4,2) \).

Problem 5. Find the volume of the smaller of the two regions enclosed by the surfaces \( z = 1 + x^2 + y^2 \) and \( x^2 + y^2 + z^2 = 11 \).

Problem 6. Show that
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = 2\pi.
\]
(Hint: the integral above is improper. To evaluate it, you should compute a triple integral over a suitably chosen bounded region and take the limit as that region grows without bound.)
Problem 7. Suppose \{u, v, w\} is a linearly independent set of vectors in a vector space \(V\). Working directly from the definition of linear independence, show that \{u + v, v + w, u + w\} is also linearly independent.

Problem 8. Let \(M\) be the 1000 \(\times\) 1000 matrix consisting of all 1s. Find the characteristic polynomial for \(M\).

Problem 9. Consider the differential equation \(y' = Ay^2\) where \(A\) is a real constant.

(a) (5 points) Find the general solution (your solution will contain the parameter \(A\)).

(b) (5 points) Find a value of \(A\) for which there exists a solution \(y(t)\) that satisfies \(y(0) > 0\) and \(y(1) < 0\) and is continuous on an open interval containing the closed interval \([0, 1]\) or explain why such an \(A\) does not exist.

Problem 10. For \(n \in \mathbb{N}\), define \(a_n = \sqrt{n+1} - \sqrt{n}\).

(a) (4 points) Compute \(\lim_{n \to \infty} a_n\).

(b) (6 points) Does the series \(\sum_{n=1}^{\infty} (-1)^n a_n\) converge absolutely, converge conditionally or diverge? Justify your answer by using one or more series tests, making sure to explain why the tests apply.