Problem 1. Show that, for an appropriate function \( y = f(x) \),
\[
\lim_{n \to \infty} \left( \frac{1}{n \sqrt{n^2 + 1^2}} + \frac{2}{n \sqrt{n^2 + 2^2}} + \cdots + \frac{n}{n \sqrt{n^2 + n^2}} \right) = \int_0^1 f(x) \, dx.
\]
Then use this fact to compute the exact value of the limit.

Problem 2. Show that the function \( f(x) = x^{1/x} \) is decreasing for all \( x \geq e \). Then use this fact to determine which of the two numbers 2006\(^{2007}\) and 2007\(^{2006}\) is larger.

Problem 3. Use an appropriate substitution to show that
\[
\int_{e^\pi}^{e^\pi + \sqrt{\pi}} (x - e - \pi)(x - e - \pi + \sqrt{\frac{2}{3}})(x - e - \pi + 2\sqrt{\frac{2}{3}}) \, dx = 1.
\]

Problem 4. Let \( (a_n) \) be a sequence of real numbers. Prove or disprove (by giving a counterexample) each of the following statements:

(a) If the series \( \sum_{n=1}^{\infty} \frac{a_n}{n} \) is convergent, then \( \lim_{n \to \infty} a_n = 0 \).

(b) If \( 0 \leq a_n \leq \frac{1}{n} \), then the series \( \sum_{n=1}^{\infty} \frac{a_n}{n} \) is convergent.

Problem 5. The operator \( S : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by \( S((x_1, x_2, x_3)^T) = (x_2, x_3, x_1)^T \).

(a) Find the matrix \( A \) of the operator \( S \) if the standard basis \( \{e_1, e_2, e_3\} \) is used for \( \mathbb{R}^3 \), where \( e_1 = (1, 0, 0)^T \), \( e_2 = (0, 1, 0)^T \), and \( e_3 = (0, 0, 1)^T \).

(b) Find the eigenvalues and eigenvectors of \( A \).
Problem 6. The $3 \times 3$ matrix $A$ satisfies the equality $AP = PB$, where

$$
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}, P = \begin{bmatrix}
1 & 0 & 0 \\
2 & -1 & 0 \\
2 & 1 & -1
\end{bmatrix}.
$$

Find $A^5$.

Problem 7. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Evaluate the double integral

$$
\int \int_D \ln(1 + x^2 + y^2) \, dx \, dy.
$$

Problem 8. Let $f(x, y) = (x^2 + y^2)e^{-x-y}$ and let $R = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ denote the first quadrant in the plane.

(a) Find the critical points of $f$ in the interior of the region $R$ and on its boundary (the positive $x$-axis and positive $y$-axis).

(b) Find the global maximum and global minimum of $f$ on the region $R$.

Problem 9. Find the general solution of the system of differential equations

$$
\begin{aligned}
\frac{dx}{dt} &= 4x + y^2 \\
\frac{dy}{dt} &= y
\end{aligned}
$$

Problem 10. Find the function $y = f(x)$ whose graph is the curve $C$ passing through the point $(2, 1)$ and satisfying the following property: each point $(x, y)$ of $C$ is the midpoint of $L(x, y)$, where $L(x, y)$ denotes the segment of the tangent line to $C$ at $(x, y)$ which lies in the first quadrant.