1. Find $y'(x)$ if $y(x) = \int_{1}^{x^2} \frac{\sin(tx)}{t} \, dt$.

**Solution:** Using the fundamental theorem of Calculus and the chain rule for functions of two variables:

$$
dx \int_{1}^{x^2} \frac{\sin(tx)}{t} \, dt = \frac{\sin(x^3)}{2} \cdot 2x + \int_{1}^{x^2} \cos(tx) \, dt = \frac{3}{x} \sin(x^3) - \frac{1}{x} \sin x.
$$

2. A solid ball has a radius of 0.5 meters and weighs 10 kg. It is floating in a large pool of water. How deeply is it submerged at its deepest point? (Recall the Archimedean principle: For a floating object, the weight of the displaced water equals the weight of the object. The density of water is approximately $1000 \text{ kg/m}^3$.)

**Solution:** If the ball is submerged to the depth $h$, the volume of the submerged portion is given by $V(h) = \int_{0.5}^{0.5-h} \int_{D_{r(z)}} dA \, dz$, where $D_{r(z)}$ is a disk centered at $(0,0)$ of radius $r(z) = \sqrt{\frac{1}{4} - z^2}$. One finds $V(h) = \pi \left( \frac{h^2}{2} - \frac{h^3}{3} \right)$.

The weight of displaced water is $1000V(h)$, so the Archimedean principle translates to the equation $1000 \pi \left( \frac{h^2}{2} - \frac{h^3}{3} \right) = 10$. The solution is $h = 0.0820$ meters.

3. Consider three tanks, labeled A, B and C. Initially, tank A contains 20 gal of water with a salt concentration of 0.2 lb/gal. Both tanks B and C initially contain 10 gal of pure water without any salt. Water is flowing from tank A to tank B at the rate of 0.1 gal/min, from tank A to tank C at the rate of 0.1 gal/min, and from tank B to tank C at the rate of 0.2 gal/min. In addition, water is leaking out of tank C at the rate of 0.1 gal/min. Assume perfect mixing, that is, the salt solution mixes effectively instantly in the tanks.

What is the salt concentration in tank C when its maximum capacity of 20 gal is reached?

**Solution:** Let $x(t)$ denote that the weight of the salt in tank C, and $y(t)$ the weight of the salt in tank B. The equation for $y(t)$ is $y' = -0.2 \frac{y}{10 - 0.1t} + 0.02$. Solving this using the method of integrating factor gives

$$
y(t) = 0.2(10 - 0.1t) - 0.02(10 - 0.1t)^2.
$$

The equation for $x(t)$ is $x' = 0.02 + 0.2 \frac{y(t)}{10 - 0.1t} - 0.1 \frac{x}{10 + 0.2t}$. Using the explicit form for $y(t)$ derived above and again the method of the integrating factor gives

$$
x(t) = 0.2(10 + 0.2t)^{3/2} - 0.004(40 - 0.2t)(10 + 0.2t)^{3/2} - 0.04 \cdot 10^{3/2}.
$$

This gives $x(t = 50) = 1.317$ lb salt in tank C after 50 minutes.
4. The temperature at any point of a flat plate is given by 

\[ T = 100 - 0.09x^2 - 0.16y^2, \]

where \(x\) and \(y\) are the vertical and horizontal distances from a fixed point \((0, 0)\), measured in feet, and \(T\) is measured in degrees Fahrenheit. Consider the point \((5, 2)\).

(a) In what direction must a bug move from \((5, 2)\) in order for temperature to decrease at the fastest rate? What is this rate (in degrees per foot)?

(b) If the bug moves at 2 ft/min in the above direction, how fast is the temperature felt by the bug decreasing (in degrees per minute)?

(c) In what direction from \((5, 2)\) must the bug move so that the temperature neither increases nor decreases?

(d) Another bug is moving along the curve of constant temperature, starting at \((5, 2)\) in the direction found in 4c. It is moving at a constant speed of 3 ft/min. Determine if the bug will return to \((5, 2)\), and if so, how long it will take.

**Solution:** We have \(\nabla T(x, y) = (-0.18x, -0.32y)\).

(a) The direction is opposite the direction of the gradient, so \(-\nabla T(5, 2) = (0.9, 0.64)\). The rate is \(|\nabla T(5, 2)| \approx 1.032 \text{ degree/ft.}\)

(b) Its \(|\nabla T(5, 2)| \cdot 2 \text{ft/min} \approx 2.065 \text{ degree/min.}\)

(c) In the direction perpendicular to \(\nabla T(x, y)\), f.ex. in the direction \((0.64, -0.9)\).

(d) It moves on the level curve \(100 - 0.9x^2 - 0.16y^2 = 100 - 0.09 \cdot 5^2 - 0.016 \cdot 2^2 \overset{\text{def}}{=} 100 - r^2\). This is an ellipse; in particular the bug will eventually return to its initial position. It can be parametrized via 

\[ x = \frac{r}{0.3} \cos t, \quad y = \frac{r}{0.4} \sin t, \quad 0 \leq t \leq 2\pi. \]

The circumference is 

\[ \ell = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2} \, dt = \int_0^{2\pi} r \sqrt{\frac{\sin^2 t}{(0.3)^2} + \frac{\cos^2 t}{(0.4)^2}} \, dt \approx 31.313 \]

The corresponding time is \(\ell/3 \text{ ft} \approx 10.44 \text{ minutes.}\)

5. Does the series \[ \sum_{n=0}^{\infty} \frac{1}{n!} \exp \left( \int_0^{2\pi} |\cos(nx)| \, dx \right) \] converge or diverge? If it converges, find its limit.

**Solution:** Compute the integral first. Using the substitution \(u = nx\) yields

\[ \int_0^{2\pi} |\cos(nx)| \, dx = \frac{1}{n} \int_0^{2n^2\pi} |\cos x| \, dx = n \int_0^{2\pi} |\cos x| \, dx = 4n. \]

Thus the series is

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \exp \left( \int_0^{2\pi} |\cos(nx)| \, dx \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \exp(4n) = \sum_{n=0}^{\infty} \frac{(e^4)^n}{n!} = e^{e^4}. \]
6. Answer the following:

(a) Define $\sum_{n=1}^{\infty} a_n$.

(b) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

(c) Find the following limits and justify your answers rigorously. (Detailed $\varepsilon - \delta$ arguments are not necessary, but clearly explain the method you used.)

(i) $\lim_{n \to \infty} \frac{1000^n}{n!}$

(ii) $\lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^{n^2}$ (Hint: Consider the cases $k > 0$, $k < 0$ and $k = 0$ separately.)

(iii) $\lim_{n \to \infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3$

**Solution:**

(a) $\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$, where $S_n = \sum_{k=1}^{n} a_k$.

(b) We have $\lim_{n \to \infty} S_{n-1} = \lim_{n \to \infty} S_n = \sum_{n=1}^{\infty} a_n$, so $0 = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = \lim_{n \to \infty} a_n$.

(c) Solutions:

(i) We know $\sum_{n=0}^{\infty} \frac{1000^n}{n!} = e^{1000}$ converges, so by (b), this implies $\lim_{n \to \infty} \frac{1000^n}{n!} = 0$.

(ii) We have by l’Hospital’s rule

$$\lim_{n \to \infty} n^2 \ln \left(1 + \frac{k}{n}\right) = \lim_{n \to \infty} \frac{\left(1 + \frac{k}{n}\right) \left(-\frac{k}{n^2}\right)}{-2/n^3}$$

$$= \lim_{n \to \infty} \frac{nk}{2} \left(1 + \frac{k}{n}\right) = \begin{cases} +\infty & \text{if } k > 0 \\ -\infty & \text{if } k < 0 \\ 0 & \text{if } k = 0 \end{cases}$$

(iii) Considering Riemann sums of width $\Delta x = 1$ (make sketch!) yields

$$\frac{1}{n^3} \int_0^n x^3 \, dx \leq \frac{1}{n^3} \sum_{k=1}^{n} k^3 \leq \frac{1}{n^3} \int_0^{n+1} x^3 \, dx,$$

so

$$\frac{1}{4} \leq \frac{1}{n^4} \sum_{k=1}^{n} k^3 \leq \frac{1}{4} \left(1 + \frac{1}{n}\right)^4.$$ 

The squeeze theorem now yields $\lim_{n \to \infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \frac{1}{4}$.

7. Answer the following:

(a) What is the interval of convergence of $\sum_{n=1}^{\infty} n x^n$?
(b) What is the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$?

(c) By differentiation or integration or some other process find the limits of the above series.

(d) How many terms are necessary to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ within an error of 0.0001?

Solution:

(a) Use the ratio test:
$$\frac{|(n+1)x^{n+1}|}{nx^n} \to |x|.$$ So the radius of convergence is 1. At $x = 1$ and $x = -1$, the series diverges (divergence test), and so the interval of convergence is $(-1, 1)$.

(b) Use again the ratio test:
$$\frac{|x^{n+1}/(n+2)|}{|x^n/(n+1)|} \to |x|.$$ So the radius of convergence is 1. At $x = 1$, the series diverges (harmonic series), and at $x = -1$, it converges (alternating series). So the interval of convergence is $[-1, 1]$.

(c) $\sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x \frac{d}{dx} \sum_{n=1}^{\infty} x^n = x \frac{d}{dx} \left( \frac{1}{1-x} - 1 \right) = \frac{x}{(1-x)^2}$.

(d) $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{x} \sum_{n=1}^{\infty} \int_0^x s^n ds = \frac{1}{x} \sum_{n=1}^{\infty} \int_0^x \left( \frac{1}{1-s} - 1 \right) ds = -\ln(1-x) + 1.$

(e) The error of approximation by using the first $N$ terms is bounded by $|a_{N+1}| = 1/(N+1)^2$, by the alternating series theorem. So we need $1/(N+1)^2 < 0.0001$, and thus $N \geq 100$.

8. Consider the function $f(x) = \frac{x^2}{1 + x^2}$ with $x > 0$. Find the point on the graph of $f$ for which the $x$-intercept of the tangent is largest.

Solution: The tangent line at the point $((x_0, f(x_0)))$ on the graph has the equation $y = \frac{x_0^2}{1 + x_0^2} + (x - x_0) \frac{2x_0}{(1 + x_0^2)^2}$. Its $x$-intercept is given by the condition $y = 0$, so
$$x = x(x_0) = \frac{1}{2} x_0 - \frac{1}{2} x_0^3.$$ The maximum of this expression (as a function of $x_0 > 0$) is attained at $x_0 = \sqrt{\frac{1}{3}}$, so the point in question is $\left(\sqrt{\frac{1}{3}}, \frac{1}{4}\right)$.

9. A square $n \times n$ matrix $A$ is called idempotent if $A^2 = A$.

(a) Show: If $\lambda$ is an eigenvalue of an idempotent matrix $A$, then $\lambda \in \{0, 1\}$.

(b) Find an example of a $2 \times 2$ idempotent matrix whose entries are all nonzero.
Show that any idempotent matrix is diagonalizable.

**Solution:**

(a) Suppose \( \lambda \) is an eigenvalue with eigenvector \( v \). Then \( Av = \lambda v \), so \( A^2v = Av = \lambda Av \), or \( (1 - \lambda)Av = 0 \). This implies that either \( 1 - \lambda = 0 \), or \( Av = 0 \). The latter statement is equivalent to \( \lambda = 0 \).

(b) It’s not hard to see that if \( A \) is an idempotent matrix, and \( S \) an invertible matrix, then \( SAS^{-1} \) is idempotent as well, i.e. similar matrices of idempotent matrices are idempotent as well. The matrix \( \text{diag}(1, 0) \) is idempotent. So for instance, the matrix

\[
A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}
\]

is idempotent.

(c) If \( A = (v_1, \ldots, v_n) \) are the columns of \( A \), then \( A^2 = A \) implies \( Av_i = v_i \) for \( i = 1, \ldots, n \). So each nonzero column vector of \( A \) is an eigenvector of the eigenvalue \( \lambda = 1 \). Hence the dimension of the eigenspace for \( \lambda = 1 \) is greater or equal to the rank of \( A \). But \( \text{rank} A + \dim \text{Ker} A = \text{rank} A + \dim \text{Eig}(A, 0) = n \), and thus the sum of the dimensions of the two eigenspaces equals the dimension of the whole vector space \( \mathbb{R}^n \). Taking bases of the two eigenspaces thus yields a linear dependent set with \( n \) elements. Thus there is a basis of eigenvectors of \( A \), and hence \( A \) is diagonalizable.

10. Consider

\[
B = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}.
\]

Here \( a \in \mathbb{R} \) is a parameter.

(a) For which values of \( a \) does \( B \) have rank 1?

(b) Find the \( n \)th power \( B^n \).

**Solution:**

(a) The characteristic polynomial is \( \det \left( \begin{array}{cc} a - \lambda & 1 \\ 1 & a - \lambda \end{array} \right) = (a - \lambda)^2 - 1 \), giving eigenvalues \( \lambda_1 = a - 1 \) and \( \lambda_2 = a + 1 \). The rank of \( B \) is 1 iff its kernel has dimension 1, which is equivalent to exactly one of the eigenvalues being zero. Hence \( B \) has rank 1 iff \( a = 1 \) or \( a = -1 \).

(b) Diagonalize \( B = S \text{diag}(a + 1, a - 1) S^{-1} \); explicitly

\[
B = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a + 1 & 0 \\ 0 & a - 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]

Hence

\[
B^n = S (\text{diag}(a + 1, a - 1))^n S^{-1} = \frac{1}{2} \begin{pmatrix} (a + 1)^n + (a - 1)^n & (a + 1)^n - (a - 1)^n \\ (a + 1)^n - (a - 1)^n & (a + 1)^n + (a - 1)^n \end{pmatrix}.
\]